The Embedding Tensor Formalism

Gauging promotes $G \subset G_0$ to a local symmetry.
Let $\{t_\alpha\}_{\alpha = 1, \dots, \dim(\mathfrak{g}_0)}$ the generators of G_0.

$$[t_\alpha, t_\beta] = \underbrace{f_{\alpha\beta}}_{\substack{\text{structure}\\ \text{constants}}}{}^{\gamma}\, t_\gamma$$

G breaks the global symmetry of the ungauged theory G_0 down to G. I will promote the structure constants of the gauge group to a tensor of G_0 to recover <u>full G_0 covariance</u>

Let's call $A^M{}_\mu$ the vectors (M in the suitable rep. of G_0 (see table 1)).

<u>Global G_0 transformations</u> (L^α, $\alpha \in \mathrm{adj}(G_0)$)

$$\delta_L A^M{}_\mu = - L^\alpha [t_\alpha]_N{}^M A^N{}_\mu \, .$$

<u>Local G transformations</u> ($\Lambda^M(x)$, $M \in V\,(\text{vec})$)

$$\delta_\Lambda A^M{}_\mu = \partial_\mu \Lambda^M \, .$$

To build D_μ by minimal coupling, we need to link $M, N \in V$ indices to $\alpha, \beta \in \mathrm{adj}(G_0)$ indices.
This is done by a <u>linear map</u>

$$\Theta : V \longrightarrow \mathfrak{g}_0$$ called <u>Embedding Tensor</u>.

$$D_\mu = \partial_\mu - \underset{\substack{\downarrow\\ \text{gauge}\\ \text{coupling}}}{g}\, A^M{}_\mu\, \Theta_M{}^\alpha\, \underbrace{t_\alpha}_{\text{generators of } G_0}$$

The generators of the gauge group G read

$$X_M = \Theta_M{}^{\alpha}\, t_{\alpha} \, .$$

The map Θ will in general transform in $V' \otimes \mathfrak{g}_0$, which is not an irrep,

$$\Theta \in V' \otimes \mathfrak{g}_0 = \vartheta_1 \oplus \vartheta_2 \oplus \ldots\ldots \oplus \vartheta_n \, ,$$

where ϑ_i's are irrep's of G_0.

Consistency Constraints

Consistency and SUSY require two sets of constraints on the embedding tensor:

i) Linear Constraint (LC)
ii) Quadratic Constraint (QC)

i) LC is a representational constraint telling me that only some ϑ_i's are allowed, whereas the other ones should be projected out. This can be derived, e.g. by imposing invariance of Θ under SUSY transformations.

ii) QC is related to the requirement of gauge-invariance for Θ.

$$\delta_{\Lambda} \Theta_N{}^{\alpha} = g\, \Lambda^M \Theta_M{}^{\beta}\left([t_{\beta}]_N{}^P \Theta_P{}^{\alpha} - f_{\beta\gamma}{}^{\alpha} \Theta_N{}^{\gamma} \right) \stackrel{!}{=} 0 \, ,$$

which yields

$$[X_M, X_N] = - X_{MN}{}^P X_P \quad , \text{ for } X_M \text{ in any repr. of } G_0$$

This immediately relates to the <u>closure of the gauge algebra</u>. Please note that

$$X_{MN}{}^{P} \equiv (X_M)_N{}^{P} \,,$$

i.e. nothing but the gauge generators in the V repr.

$$\mathrm{QC} \begin{cases} [MN] & \text{Jacobi identities} \\ (MN) & \text{anti-symmetry of the brackets} \end{cases}$$

The "generalised structure constants" $X_{MN}{}^{P}$ <u>don't need to be antisymmetric by themselves</u>, i.e. $X_{(MN)}{}^{P} \neq 0$ in general. Only @ quadratic level

$$X_{(MN)}{}^{P} (X_P)_{\text{any rep.}} \overset{!}{=} 0 \,,$$

and in particular when X_P is given again in V

$$X_{(MN)}{}^{P} X_{PQ}{}^{R} = 0 \,,$$

which is a QC in $X_{MN}{}^{P}$.

$\boxed{\text{Summarising}}$ (Schematically)

Any Θ satisfying

$$\mathbb{P}_1(\Theta) = 0 \quad (\mathrm{LC}) \,,$$
$$\mathbb{P}_2(\Theta \otimes \Theta) = 0 \quad (\mathrm{QC}) \,,$$

parametrises a <u>consistent gauging</u> of my supergravity.

D	G_0	H	# scalars	V	Θ
9	$\mathbb{R}^+ \times SL(2)$	$SO(2)$	3	$1_{(+4)} \oplus 2_{(-3)}$	$2_{(+3)} \oplus 3_{(-4)}$
8	$SL(2) \times SL(3)$	$SO(2) \times SO(3)$	7	$(2, 3')$	$(2,3) \oplus (2,6')$
7	$SL(5)$	$SO(5)$	14	$10'$	$15 \oplus 40'$
6	$SO(5,5)$	$SO(5) \times SO(5)$	25	16	144_s
5	$E_{6(6)}$	$USp(8)$	42	$27'$	351
4	$E_{7(7)}$	$SU(8)$	70	56	912

Table 1 (Maximal Supergravities)

Example of LC $D = 9$ $G_0 = \mathbb{R}^+ \times SL(2)$

$$\mathrm{adj}(G_0) = \mathfrak{g}_0 = 1_{(0)} \oplus 3_{(0)}$$

$$V = 1_{(+4)} \oplus 2_{(-3)} \longrightarrow V' = 1_{(-4)} \oplus 2_{(+3)}$$

$$\Theta \in \mathrm{adj}(G_0) \otimes V = \left(1_{(0)} \oplus 3_{(0)}\right) \otimes \left(1_{(-4)} \oplus 2_{(+3)}\right) =$$

$$= \cancel{1_{(-4)}} \oplus 3_{(-4)} \oplus 2_{(+3)} \oplus \cancel{2_{(+3)}} \oplus \cancel{4_{(+3)}}$$

$\checkmark$ $\checkmark$

In this case $\Theta = \vartheta_1 \oplus \vartheta_2$, where

$$\vartheta_1 \equiv \kappa^{(ij)} \in 3_{(-4)}, \quad \vartheta_2 \equiv \vartheta^i \in 2_{(+3)} .$$

Then one can work out the QC, which read

$$\begin{cases} \kappa^{(ij}\,\vartheta^{k)} = 0, \\ \kappa^{ij}\,\vartheta^{k}\,\varepsilon_{jk} = 0. \end{cases}$$

Relation to the Kac – Moody approach

M-theory $\qquad\qquad$ Max Sugra

$11 \longrightarrow 11-d$

$\uparrow$

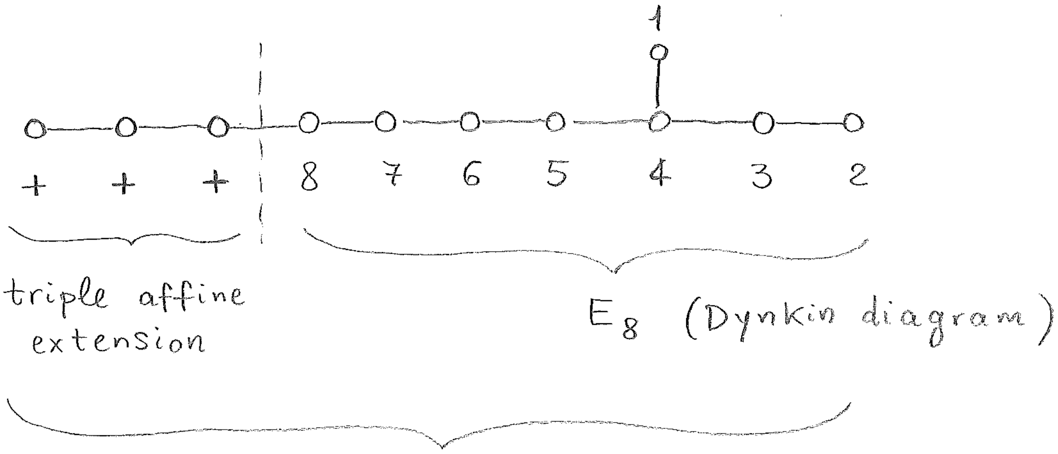
Naïvely expect $SL(d)$ duality symmetry

Hidden Symmetry : $G_0 \supset SL(d)$ (usually $E_{d(d)}$)

Imagine now to take $d = 11 \to$ M-theory on T^{11}

This yields a 0-dimensional theory that realises a Kac-Moody extended symmetry E_{11}.

↳ ∞-dimensional algebra obtained as $E_{11} = E_8^{+++}$

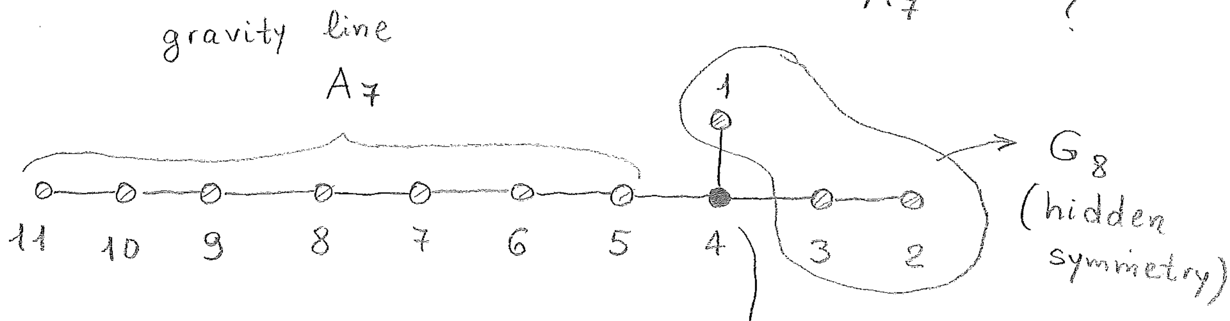


$$E_{11} = E_8^{+++}$$

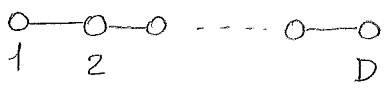
Breaking E_{11} into $\underbrace{A_{D-1}}_{SL(D)} \times \underbrace{G_D}_{?}$ gives exactly the duality symmetry of maximal supergravity in D dimensions, $0 < D < 11$.

Example : D = 8 maximal supergravity

I need to break $E_{11} \longrightarrow \underbrace{SL(8)}_{A_7} \times \underbrace{G_8}_{?}$



inactivated node
(direct factorisation spacetime ⊗ internal!)

($A_D \equiv$ )

So I find

$$G_8 = \circ \quad \circ\!-\!\!-\!\circ \quad = A_1 \times A_2 = SL(2) \times SL(3)$$

→ Try to see the origin of type IIA & IIB supergravities as two topologically inequivalent ways of choosing the gravity line representing A_9 inside E_{11}.

By means of this prescription, one can derive the global symmetry G_0 of all maximal supergravities and determine their content in terms of fields and deformations (Θ) arranged into irrep's of G_0. In particular it gives the solutions to the LC (i.e. the allowed G_0 irrep's for Θ)

→ This can be done for half-maximal supergravities as well, and it has already been successful. However, since they can be coupled to an arbitrary number N of vector multiplets, every N needs a different Kac-Moody algebra.

For $N = 10 - D$, D_8^{+++} is the working one ($SO(8,8)^{+++}$)

Fermions and Supersymmetry

Scalars parametrise a coset $\frac{G_0}{H}$, where G_0 is the global symmetry and H its maximal compact subgroup. This has vielbein reprentative $\mathcal{V}$ where global G_0 and local H transformations act respectively from the left and left on $\mathcal{V}$, i.e.

$$\mathcal{V} \longmapsto L \, \mathcal{V} \, h(x) \, ,$$

where $L \in G_0$, $h(x) \in H$.
The role of the vielbein is crucial to construct couplings between bosons (G_0 irrep's) and fermions (H irrep's)

The gauging procedure works as

$$\mathcal{L}_{\text{gauged}} = \mathcal{L}_{\text{ungauged}} [\partial \rightarrow D] + \underbrace{\underset{\substack{\downarrow \\ \text{gauge-cov.} \\ \text{topological} \\ \text{terms}}}{\mathcal{L}_{\text{top}}} + \underset{\substack{\downarrow \\ \text{couplings} \\ \text{scalars-} \\ \text{fermions}}}{\mathcal{L}_{\text{fermi mass}}} + \underset{\substack{\downarrow \\ \text{Scalar} \\ \text{potential}}}{\mathcal{L}_{\text{pot}}}}_{\text{all these are needed to restore SUSY!}}$$

D	H	H_R	ψ_μ	λ
9	$SO(2)$	$U(1)$	$1_{(+1)} \oplus 1_{(-1)}$	$2 \cdot 1_{(+1)} \oplus 2 \cdot 1_{(-1)}$
8	$SO(2) \times SO(3)$	$U(2)$	$2_{(+1)} \oplus \overline{2}_{(-1)}$	$2_{(+1)} \oplus \overline{2}_{(-1)} \oplus 4_{(+3)} \oplus \overline{4}_{(-3)}$
7	$SO(5)$	$USp(4)$	4	16
6	$SO(5) \times SO(5)$	$USp(4) \times USp(4)$	$(1,4) \oplus (4,1)$	$(5,4) \oplus (4,5)$
5	$USp(8)$	$USp(8)$	8	48
4	$SU(8)$	$SU(8)$	$8 \oplus \overline{8}$	$56 \oplus \overline{56}$

$$(\delta\psi_\mu)_{gauged} = (\delta\psi_\mu)_{ungauged} + g\,\mathcal{A}_1\,\varepsilon\,,$$

$$(\delta\lambda)_{gauged} = (\delta\lambda)_{ungauged} + g\,\mathcal{A}_2\,\varepsilon\,,$$

(new SUSY rules)

and

$$e^{-1}\mathcal{L}_{\text{fermi mass}} = g\Big[\mathcal{A}_1\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + (\mathcal{A}_2\bar{\psi}_\mu\gamma^\mu\lambda + h.c.) +$$

$$+ (\mathcal{A}_3\bar{\lambda}\lambda + h.c.)\Big]$$

$\hookrightarrow$ dependent on $\mathcal{A}_1$ & $\mathcal{A}_2$

$\mathcal{A}_1$ & $\mathcal{A}_2$ are the irreducible components of Θ w.r.t. H (often called T-tensor), obtained by contracting Θ (carrying G_0 indices) w/ copies $\mathcal{V}$ (transforming G_0 into H indices).
$\to$ directly talking to the fermions!

D	$H = H_R$	T-tensor
9	$U(1)$	$1_{(-2)} \oplus 1_{(-1)} \oplus 1_{(0)} \oplus 1_{(+1)} \oplus 1_{(+2)}$
8	$U(2)$	$1_{(-1)} \oplus 1_{(+1)} \oplus 3_{(-1)} \oplus 3_{(+1)} \oplus 5_{(-1)} \oplus 5_{(+1)}$
7	$USp(4)$	$1 \oplus 5 \oplus 14 \oplus 35$
6	$USp(4) \times USp(4)$	$(4,4) \oplus (4,16) \oplus (16,4)$
5	$USp(8)$	$36 \oplus 315$
4	$SU(8)$	$36 \oplus \overline{36} \oplus 420 \oplus \overline{420}$

Scalar potential

$$e^{-1}\mathcal{L}_{pot} = g^2\Big(\underbrace{|\mathcal{A}_1|^2}_{\substack{\text{gravitino}\\\text{mass}\\\text{(AdS)}}} - \underbrace{|\mathcal{A}_2|^2}_{\substack{\text{dilatino}\\\text{mass}\\\text{(}\cancel{\text{SUSY}}\text{)}}}\Big) = -V$$